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Entanglement generation from deformed spin coherent states using a beam splitter

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Abstract
Using the linear entropy as a measure of entanglement, we investigate the effect of a beam splitter on the Perelomov coherent states for the q-deformed $U_q(su(2))$ algebra. We distinguish two cases: in the classical $q \to 1$ limit, we find that the states become Glauber coherent states as the spin tends to infinity; whereas for $q \neq 1$, the states, contrary to the earlier case, become entangled as they pass through a beam splitter. The entanglement strongly depends on the q-deformation parameter and the amplitude Z of the state.

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1. Introduction

Quantum entanglement is one of the most important manifestations which distinguishes the quantum world from the classical counterpart. As a physical resource, it plays a vital role in various fields of quantum information theory, such as quantum cryptography [1–3], quantum computation [4], quantum teleportation [5], etc. Therefore, the characterization and the quantification of entanglement have attracted much attention and became one of the most studied problems in recent years. In order to quantify entanglement a number of measures have been proposed, such as concurrence [6, 7], entanglement of formation [8, 9] and linear entropy [10, 11]. The fundamental problem for entanglement is to test whether a given state of a composite quantum system, consisting of two or more subsystems, is separable or not; it is entangled if it cannot be decomposed into a direct product of the states of the subsystems.

The preparation of entangled states has been studied extensively and constitutes an essential step in quantum information theory. Recently, numerous devices have been proposed and realized experimentally to generate quantum entanglement, such as beam splitter [12–18], cavity QED [19, 20], NMR systems [21, 22], etc. A beam splitter is a linear optical device used to generate quantum entanglement between two modes [23]. Its effect, as a lossless four-port device, is mathematically described by a unitary transformation connecting the input fields and the output fields. Kim *et al* [16] studied the entangling properties of a beam splitter with different input states, such as Fock states, coherent states, squeezed states and Gaussian mixed states. They conjectured that in order to obtain entangled output states of a beam splitter, a necessary condition is that at least one of the input fields should be nonclassical; this was later proved in [24].

Another important concept that has attracted much attention in quantum information theory is the notion of coherent, or quasi-classical states. These states are very useful for investigating different problems in quantum physics [25–27] and have diverse applications in many branches of physics [28, 29]. The coherent states were first introduced by Schrödinger [30] in 1926, in the context of the harmonic oscillator and have been extensively studied in physics [28, 31]. In 1965, the harmonic oscillator coherent states became very important in quantum optics due to the seminal work of Glauber [32]. As eigenstate of Bose annihilation operator $\widehat{a}$ ($\widehat{a}|\alpha\rangle = \alpha|\alpha\rangle$), he realized that these states have the interesting property of minimizing the Heisenberg uncertainty relation. In 1972, Peremolov introduced the spin coherent states or $SU(2)$ coherent states [33, 34] which are associated with the $SU(2)$ group. These states describe several systems and have many applications in quantum optics, statistical mechanics and condensed matter physics [28, 31].

On the other hand, the quantum groups were introduced as a mathematical description of deformed Lie algebra that gave the possibility to construct deformed coherent states. They were introduced as a natural extension of the notion of coherent states [35, 36]. Generalized deformation of Glauber states were constructed, see [37], as related to deformed harmonic oscillators. Deformed spin coherent states were also constructed as coherent states related to the quantum algebra $su_q(2)$ [38, 39].

The physical importance of the deformed coherent states lies in the fact that they offer a best description for non-ideal physical devices such as lasers (i.e. real lasers) [40]. The deformation parameter plays then the role of a tuning parameter defining how far the realized device is from the ideal one.

The aim of the present paper is to investigate in detail the entanglement generated via a beam splitter when a deformed spin coherent state is injected into one port and the vacuum state is injected into the other. We show that this entanglement depends on the q-deformation parameter and the amplitude, Z, of the state. For $q = 1$, the deformed spin coherent state becomes the ordinary spin coherent state and one recovers the results of Markham and Vedral in [41]. Namely, the entanglement tends to disappear in the limit of indefinitely high spin. This, in turn, is due to the fact that in this limit the spin coherent states become Glauber coherent states. A behaviour completely different arises when $q \neq 1$; the output state is then always entangled such that the entanglement passes through a minimum depending on the value of the q-deformation parameter and increases thereafter to maintain a very slow growth.

The organization of this paper is as follows: section 2 is devoted to the construction of Perelomov coherent states for the deformed $su(2)$ algebra and their representation in Fock space. In section 3, we examine the entanglement as the result of the effect of a beam splitter on the deformed spin coherent state using the linear entropy. In section 4, we present summary of our results.

2. Deformed spin coherent states and Fock space

The general bases for building up a quantum group (which is a deformation of a corresponding 'usual' group or algebra) were independently given in [38, 39, 42], using two different approaches.

In this paper we are interested in the deformation of the $su(2)$ algebra which we denote as $U_q(su(2))$, since it is in fact a deformation of the enveloping algebra $U(su(2))$.

The quantum algebra (deformed algebra) $U_q(su(2))$ is generated by three generators $J_\pm^q$ and J_z^q obeying the following commutation relations,

$$\left[J_z^q, J_\pm^q\right] = \pm J_\pm^q, \qquad \left[J_+^q, J_-^q\right] = \left[2J_z^q\right]_q, \tag{1}$$

where, q is a real parameter (although one can consider complex values for q, in this paper we will focus on the results for $q \in R$) and where the 'box function' is defined by

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sinh(\gamma x)}{\sinh(\gamma)} \qquad \text{for} \qquad q = e^\gamma \quad \gamma \in R.$$

An important property is that one recovers the undeformed algebra $su(2)$ by taking the limit $q \longrightarrow 1$(or $\gamma \longrightarrow 0$).

The generators obey also the following relations:

$$\left(J_+^q\right)^+ = J_-^q, \qquad \left(J_-^q\right)^+ = J_+^q, \qquad \left(J_z^q\right)^+ = J_z^q. \tag{2}$$

The unitary irreducible representation of the $U_q(su(2))$ are similar to those of $su(2)$; they are indexed using a single positive and half-interger parameter j. The orthonormal basis of the space of representation is denoted as, $|j, m\rangle$, with $m = j, j-1, \ldots, -j$. The generators act on this basis following the rules,

$$\begin{aligned} J_z^q|j,m\rangle &= m|j,m\rangle, \\ J_\pm^q|j,m\rangle &= ([j \mp m]_q[j \pm m + 1]_q)^{\frac{1}{2}}|j, m \pm 1\rangle. \end{aligned} \tag{3}$$

The deformed spin coherent sates are coherent sates that are constructed using a formally analogous scheme as the one allowing the construction of the spin coherent states starting from the algebra $su(2)$ [35, 36]. In this way, and in a given j-representation of $U_q(su(2))$ these states are defined by

$$|Z, j\rangle_q = \mathcal{N}(|Z|^2) e_q^{ZJ_+^q}|j, -j\rangle, \qquad Z \in \mathcal{C}, \tag{4}$$

where we introduced the deformed exponential function

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \qquad \text{with} \qquad [n]_q! = [n]_q[n-1]_q \cdots [1]_q \qquad \text{and} \qquad [0]_q! = 1.$$

The normalization factor takes the shape

$$\mathcal{N}(|Z|^2) = \frac{1}{\sqrt{(1 + |Z|^2)_q^{2j}}},$$

where, the q-binomial formula (deformed version of Newton's binomial formula) [43] is used

$$(x + y)_q^n := \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q x^{x-m} y^m = \prod_{k=1}^{n} (x + q^{n-2k+1} y). \tag{5}$$

Here, the q-binomial function is

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[n]_q![n-m]_q!} \qquad \text{for} \qquad n \geqslant m. \tag{6}$$

Using these definitions, we may write the deformed spin coherent states as

$$|Z, j\rangle_q = \left((1 + |Z|^2)_q^{2j}\right)^{-\frac{1}{2}} \sum_{m=-j}^{j} \left(\begin{bmatrix} 2j \\ j+m \end{bmatrix}_q\right)^{\frac{1}{2}} Z^{(j+m)}|j, m\rangle. \tag{7}$$

Of course, for the particular value $q = 1$, one recovers the usual spin coherent states [34].

In a manner similar to [41, 44], and in order to apply these states in the current context one needs to express the basis vectors $|j, m\rangle$ in terms of the Fock states $|n\rangle$ ($|j, m\rangle \sim |n\rangle$). In [41], this was done using the Holstein–Primakoff realization of the $su(2)$ algebra.

In our case, $U_q(su(2))$, one can achieve the same result using an alteration of the later realization:

$$J_+^q = a_q^\dagger \sqrt{[2j - N]_q}, \qquad J_-^q = \sqrt{[2j - N]_q} a_q, \qquad J_z^q = N - j, \tag{8}$$

where $a_q, a_q^\dagger$ are deformed annihilation and creation operators acting on the Fock states, such that

$$a_q|n\rangle = \sqrt{[n]_q}\,|n-1\rangle, \qquad a_q^\dagger|n\rangle = \sqrt{[n+1]_q}|n+1\rangle, \qquad N|n\rangle = n|n\rangle. \tag{9}$$

They obey the following relations [45, 46]:

$$a_q a_q^\dagger - q a_q^\dagger a_q = q^{-N}, \qquad [N, a_q] = -a_q, \qquad \left[N, a_q^\dagger\right] = a_q^\dagger. \tag{10}$$

Using this realization and especially the last one given in equation (8), one gets the following change of variables $m = n - j$ or $n = j + m$, which when applied in equation (7) yields the following expression of the deformed spin coherent states in terms of the Fock states

$$|Z, j\rangle_q = \left((1 + |Z|^2)_q^{2j}\right)^{-\frac{1}{2}} \sum_{n=0}^{2j} \left(\begin{bmatrix} 2j \\ n \end{bmatrix}_q \right)^{\frac{1}{2}} Z^n |n\rangle. \tag{11}$$

3. Action of a beam splitter

We describe the effect of a beam splitter on a state, $|\psi\rangle$, as it passes through one port, while the other port is in the vacuum state. We assume that our beam splitter is 50:50 and the reflected beam suffers a phase shift of $\frac{\pi}{2}$. We consider that the horizontal input beam contains the state that interests us while the vacuum state is always in the vertical input beam (see figure 1). The action of the beam splitter can be described by a unitary operator $\widehat{U}_{\mathrm{BS}}$ that relates the input state to the corresponding output state by

$$|\mathrm{out}\rangle = \widehat{U}_{\mathrm{BS}}|\mathrm{int}\rangle, \tag{12}$$

where the unitary operator $\widehat{U}_{\mathrm{BS}}$ is [16]

$$\widehat{U}_{\mathrm{BS}} = \mathrm{e}^{\mathrm{i}\frac{\theta}{2}(a^\dagger b + a b^\dagger)}. \tag{13}$$

Here a and b (respectively $a^\dagger$ and $b^\dagger$) are the annihilation operators (respectively creation operators) describing the input fields.

Let us first introduce the effect of a 50:50 beam splitter on an input state comprised of a number state in the horizontal input beam, that is $|\psi\rangle = |n\rangle$, and vacuum state $|0\rangle$ in the vertical input beam

$$\widehat{U}_{\mathrm{BS}}|n\rangle|0\rangle = \sum_{p=0}^{n} \binom{n}{p}^{\frac{1}{2}} T^p R^{(n-p)} |p\rangle|n-p\rangle, \tag{14}$$

where, T and R are, respectively, the complex transition and reflexion coefficients satisfying the normalization condition $|T|^2 + |R|^2 = 1$. As mentioned previously, the beam splitter is 50:50 and the reflected beam receives a phase shift of $\frac{\pi}{2}$, then we have $T = \frac{1}{\sqrt{2}}$ and $R = \frac{\mathrm{i}}{\sqrt{2}}$. In this case equation (14) becomes

$$U_{\mathrm{BS}}|n\rangle|0\rangle = \sum_{p=0}^{n} \binom{n}{p}^{\frac{1}{2}} \left(\frac{1}{\sqrt{2}}\right)^p \left(\frac{\mathrm{i}}{\sqrt{2}}\right)^{(n-p)} |p\rangle|n-p\rangle. \tag{15}$$

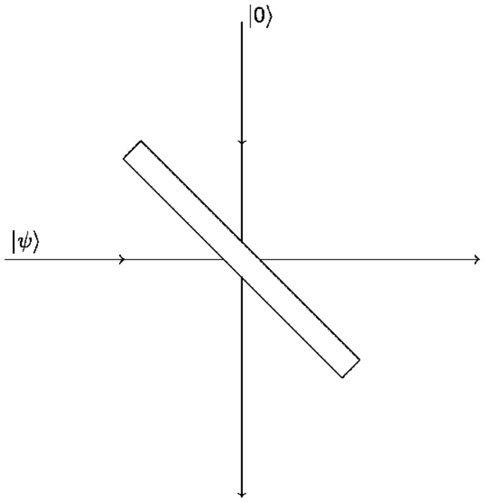

Figure 1. A 50:50 beam splitter with a state $|\psi\rangle$ on the horizontal port and a vacuum state $|0\rangle$ on the vertical port.

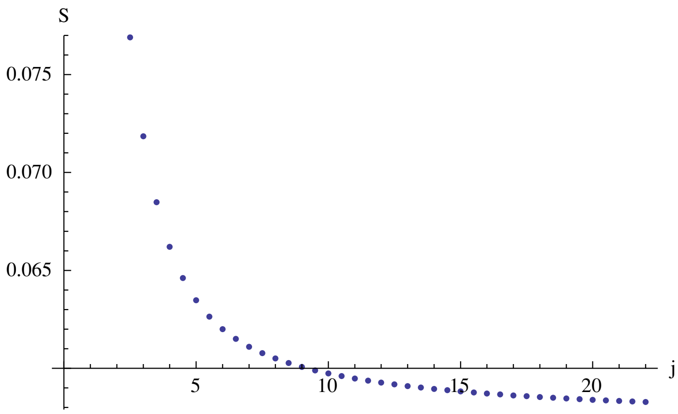

Figure 2. The linear entropy as a function of the spin j for $q = 1$ and $|Z| = 1$.

Here, p and $n - p$ characterize the basis states of the output ports.

It can be easily verified that when the horizontal input contains a Glauber state [32], defined as $|\alpha\rangle = \mathrm{e}^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n}} |n\rangle$, and the vacuum state $|0\rangle$ in the other port, the output state from the beam splitter is

$$\begin{aligned} |\mathrm{out}\rangle &= \widehat{U}_{\mathrm{BS}} |\alpha, 0\rangle \\ &= e^{-\frac{|\alpha|}{2}^2} \sum_{p=0}^{\infty} \frac{\alpha^p}{\sqrt{p!}} T^p |p\rangle \otimes \sum_{(n-p)=0}^{\infty} \frac{\alpha^{(n-p)}}{\sqrt{(n-p)!}} R^{(n-p)} |n - p\rangle \\ &= \left| \frac{\alpha}{\sqrt{2}} \right\rangle \otimes \left| \mathrm{i} \frac{\alpha}{\sqrt{2}} \right\rangle, \end{aligned} \tag{16}$$

which is a product of two coherent states with zero entanglement.

Recently, Markham and Vedral [41] investigated the effect of a beam splitter on the spin coherent states or Perelomov coherent states of $su(2)$ Lie algebra for a single mode field.

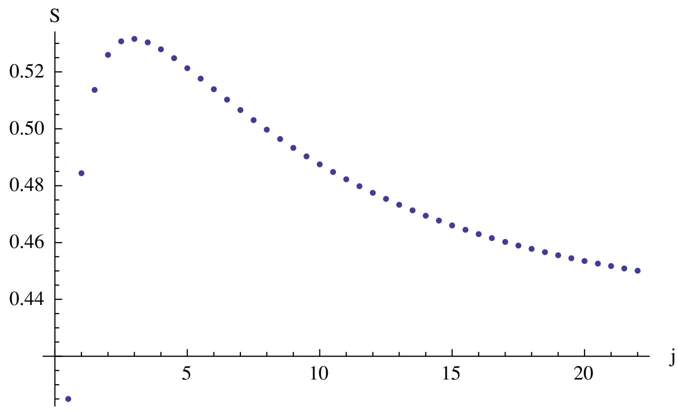

Figure 3. The linear entropy as a function of the spin j for $q = 1$ and $|Z| = 3$.

They show that these states approach to Glauber states as the spin tends to infinity, and thus to become separable states (product of coherent states) after they pass the beam splitter and, thus, have zero entanglement.

Let us now study the entanglement of deformed spin coherent state after being passed through a beam splitter. We analyse and measure the entanglement of the beam splitter output state by using the linear entropy. In fact, the effect of a beam splitter on an input state comprised of a deformed spin coherent state in one mode and the vacuum state in the other is obtained using equations (11) and (14) and is given by

$$|\mathrm{out}\rangle = \widehat{U}_{\mathrm{BS}}|Z, j\rangle_q|0\rangle$$
$$= \frac{1}{\left((1+|Z|^2)_q^{2j}\right)^{\frac{1}{2}}}\sum_{n=0}^{2j}\sum_{p=0}^{n}\left(\begin{bmatrix}2j\\ n\end{bmatrix}_q\right)^{\frac{1}{2}}\begin{pmatrix}n\\ p\end{pmatrix}^{\frac{1}{2}} Z^n T^p R^{(n-p)}|p\rangle|n-p\rangle \tag{17}$$

where

$$T = \frac{1}{\sqrt{2}}, \qquad R = \frac{\mathrm{i}}{\sqrt{2}}.$$

As a measure of entanglement, we use the linear entropy (upper bound of the Van Neumann entropy), which is reasonable in several senses as an entanglement monotone because it gives a bound on the entropy entanglement and as a monotonic function of the Schmidt values [10, 41]. It is defined for the bipartite system as

$$S = 1 - \mathrm{Tr}\left(\rho_A^2\right), \tag{18}$$

where ρ_A is the reduced density operator for a system A. It is obtained by performing a partial trace over system B of the density matrix for the combined system ρ_{AB}. In the present case, from equation (17) we obtain

$$\rho_A = \frac{[2j]_q!}{(1+|Z|^2)_q^{2j}}\sum_{p,p'}^{2j}\sum_{m=0}^{\min(2j-p,2j-p\prime)} \times\left(\frac{(m+p)!(m+p')!}{[m+p]_q![2j-m-p]_q![m+p']_q![2j-m-p']_q!}\right)^{\frac{1}{2}} \times\frac{|Z|^{2m}}{m!}|T|^{2m}\frac{Z^p\overline{Z^{p'}}}{\sqrt{p!p'!}}R^p\overline{R^{p'}}|p\rangle\langle p'|. \tag{19}$$

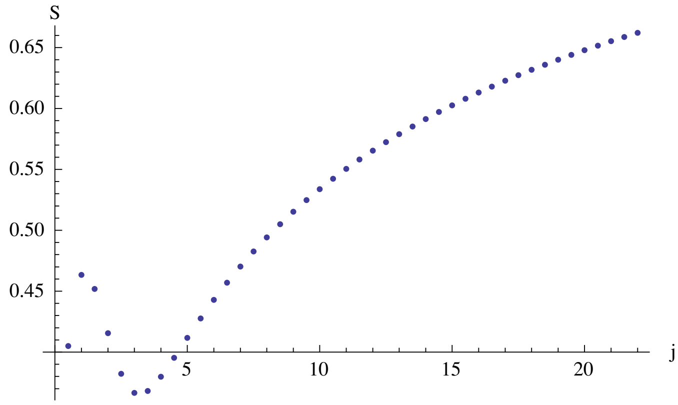

Figure 4. The linear entropy as a function of the spin j for $q = 0.5$ and $|Z| = 3$.

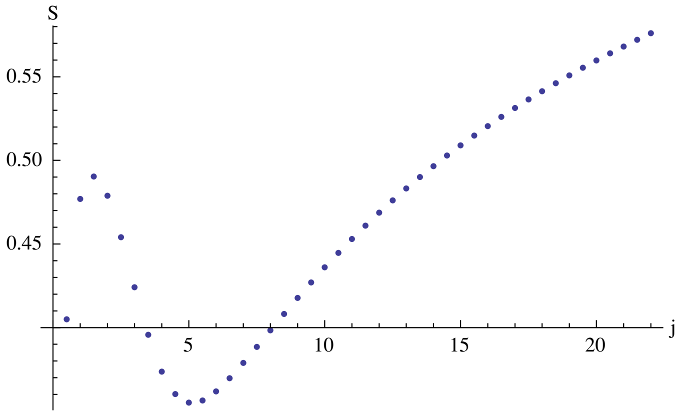

Figure 5. The linear entropy as a function of the spin j for $q = 1.5$ and $|Z| = 3$.

Hence, the linear entropy is written as

$$S = 1 - \left(\frac{[2j]_q!}{(1+|Z|^2)_q^{2j}} \right)^2 \sum_{p,p'}^{2j} \sum_{m,m'=0}^{\min(2j-p,2j-p\prime)} \times \left(\frac{L_{m,m'}^{p,p'}}{M_{m,m'}^{p,p'} N_{m,m'}^{p,p'}} \right)^{\frac{1}{2}} \frac{|Z|^{2(m+m'+p+p')}}{m!m'!p!p'!} |T|^{2(m+m')} |R|^{2(p+p')}, \tag{20}$$

where

$$L_{m,m'}^{p,p'} = (m+p)!(m+p')!(m'+p)!(m'+p')!,$$
$$M_{m,m'}^{p,p'} = [m+p]_q![m+p']_q![m'+p]_q![m'+p']_q!,$$
$$N_{m,m'}^{p,p'} = [2j-m-p]_q![2j-m-p']_q![2j-m'-p]_q![2j-m'-p']_q!.$$

It is remarkable to note that, for $q = 1$ this expression is equal to that obtained earlier in [41]. In fact, as for $q = 1$ the deformed spin coherent states become the ordinary spin coherent states which is an expected result.

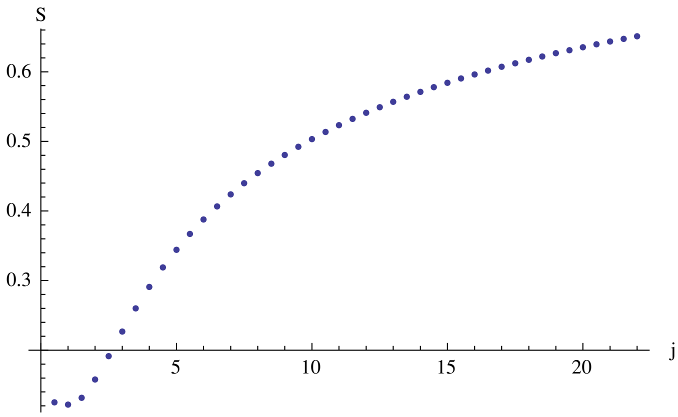

Figure 6. The linear entropy as a function of the spin j for $q = 0.5$ and $|Z| = 1$.

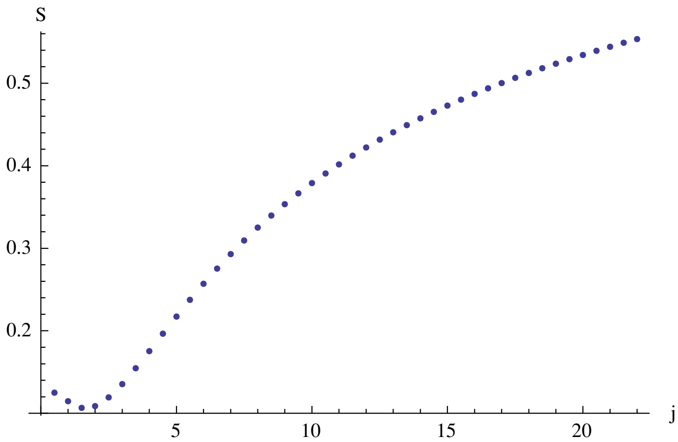

Figure 7. The linear entropy as a function of the spin j for $q = 1.5$ and $|Z| = 1$.

The main result obtained in [41] is that the entanglement is highly dependent on the value of Z. It is also shown that for significantly high values of the spin j the linear entropy tends to zero, meaning that the entanglement tends to disappear for these values. However this decrease of the linear entropy is too slow and it is difficult to achieve these high values of spin experimentally. The Z dependence in [41] can be summarized as follows: when $|Z| \leqslant 1$ the linear entropy decreases from the maximum, attained at the origin, to a value after which it tends slowly to zero (see figure 2). For $|Z| > 1$ it increases to a maximum then decreases and declines slowly to zero thereafter (see figure 3). For the formula of the linear entropy we obtained (20), a comparison should first be addressed with the results of [41] (i.e., for ordinary spin coherent states or equivalently for $q = 1$).

For $|Z| = 3$ the linear entropy obtained in equation (20) is subjected to an initial quick change strongly depending on the amplitude Z: it goes to a maximum, declines to a minimum corresponding to the minimal entanglement of the output state and grows again. The linear entropy then settles to a very slow increasing (see figures 4 and 5). For a given value of Z, the minimum value of entanglement is dependent on q. In fact the closer q is to 1 (from left

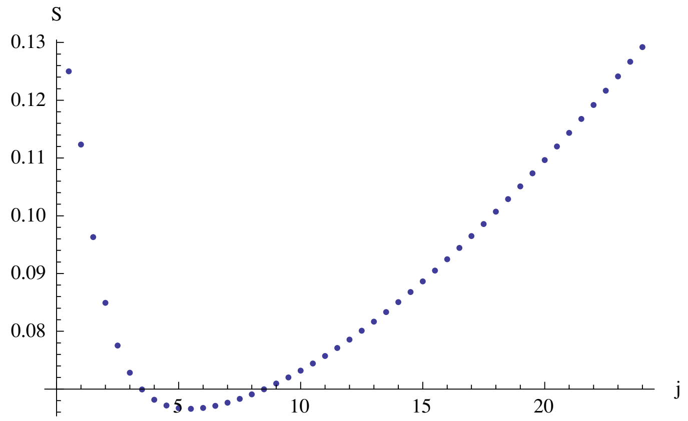

Figure 8. The linear entropy as a function of the spin j for $q = 0.95$ and $|Z| = 1$.

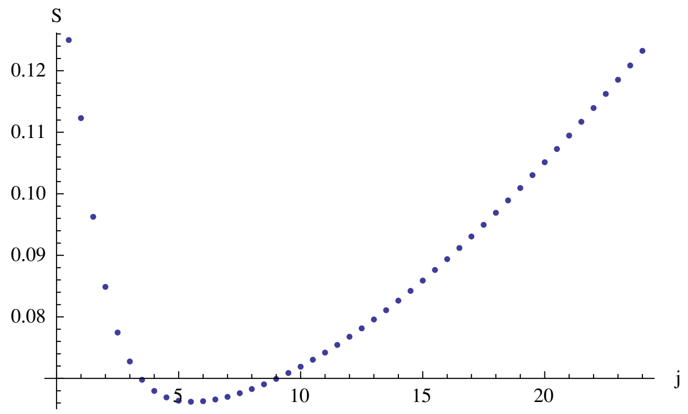

Figure 9. The linear entropy as a function of the spin j for $q = 1.05$ and $|Z| = 1$.

or right), the higher the value of j for which the minimum is attained. For the limiting value $q = 1$ this value tends to infinity.

When $|Z| = 1$ we can note a similar classification. For $q \neq 1$, the linear entropy declines to a minimum (minimal entanglement of the output state, which depends on the value of q) and increases thereafter to maintain a very slow growth. In case the values of q are close to one, the minimum is attainted for greater spin (see figures 6–9). This minimum is reached only at infinity for $q = 1$. Another remarkable feature, figure 11, is that the minimum value of entanglement gets smaller as q approaches the value $q = 1$ (from left or light), the limiting value being zero entanglement but it is achieved for infinitely large values of j and $q = 1$.

We note that when $|Z| = 3$ and q is very close to 1, the minimum of the linear entropy is attained for a spin value j which is larger than the case when $|Z| = 1$ (for example, when $|Z| = 3$ and $q = 0.95$ the minimum of entanglement is attained at $j = 34.5$ whereas in the case where $|Z| = 1$ the minimum is at $j = 5$ (figure 8)).

However, for every value of the amplitude Z, the q-deformation parameter does not have any impact on the linear entropy for $j = \frac{1}{2}$, then the output state has the same initial entanglement. From another side, when we are concerned with small amplitude, the entanglement of the output state is close to zero for $q = 1$. When the q-deformation

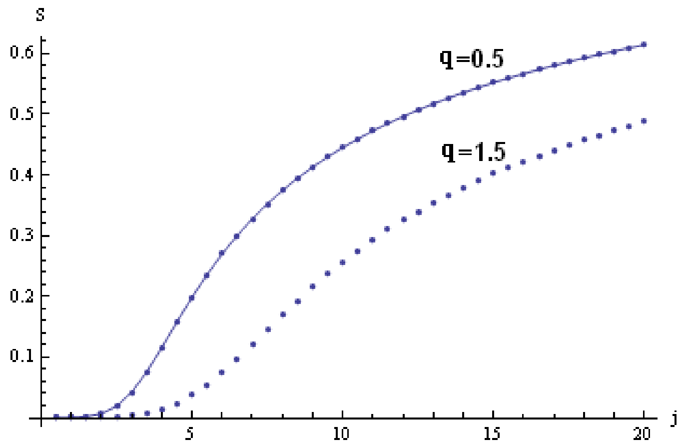

Figure 10. The linear entropy as a function of the spin j for $|Z| = 0.2$.

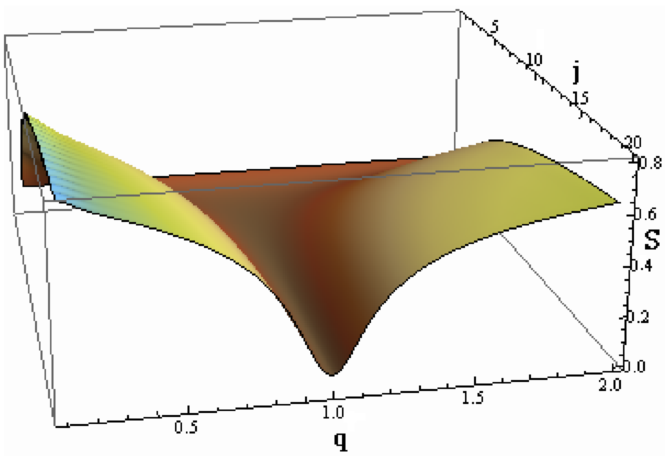

Figure 11. The linear entropy as a function of spins j and q for $|Z| = 1$.

parameter varies around one, the linear entropy vanishes approximatively only for small spin j (see figure 10).

We stressed earlier that for $|Z| \leqslant 1$ (figure 11) the entanglement gets smaller as the deformation parameter q gets closer to 1 (either from above or below). This behaviour changes for $|Z| > 1$ in fact plotting the linear entropy versus q (for $|Z| = 3$ and different values of j), figures 12 and 13 show that the entanglement decreases and attains a local minima, whereas it latter rises to a local maxima for $q = 1$. The value of the minimum entanglement depends on the spin j (and of course on Z also). In fact, higher the value of the spin, higher the value of q at which the minimum entanglement occurs (provided that $q \leqslant 1$; however, if one considers $q \geqslant 1$ then this value gets smaller as j gets higher).

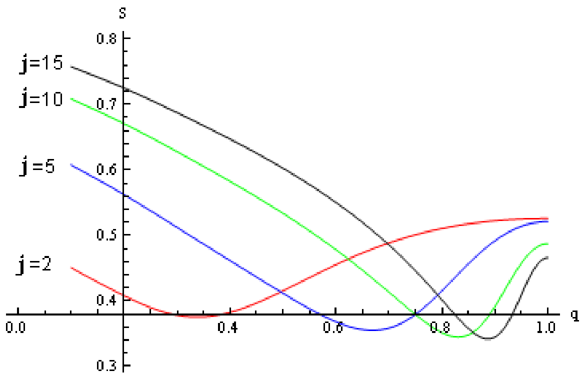

Figure 12. The linear entropy as a function of q for $|Z| = 3$.

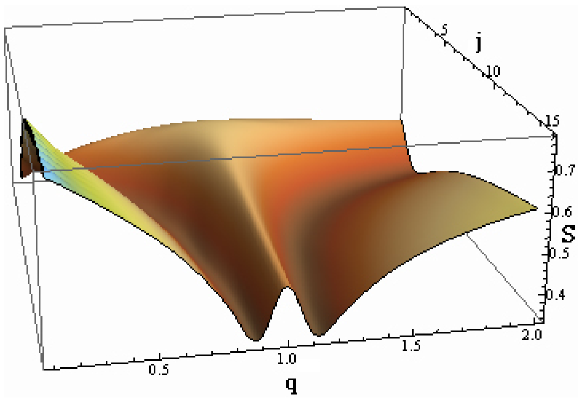

Figure 13. The linear entropy as a function of spins j and q for $|Z| = 3$.

4. Conclusion

In summary, we have introduced the Perelomov coherent states for the $U_q(su(2))$ quantum algebra and their representation in Fock states. Using the linear entropy as a measure of entanglement, we have investigated the entanglement generated via a beam splitter when the deformed spin coherent state is incident on one input port and the vacuum state is incident on the other.

We show that the entanglement of the output state depends on the q-deformation parameter and the amplitude Z of the state. For each value of $|Z|$, the entropy is subjected to an initial quick change and thereafter settles to a very slow monotony. The entanglement has a minimum value depending on the value of q: for q closer to 1 (from left or right) this minimal entanglement is attained for larger spin (for $q = 1$ the minimum vanishes for higher spin).

We also note that for $|Z| \leqslant 1$, the entanglement of the output state decreases as the q-deformation parameter approaches to 1 for all spin j and it is minimal at $q = 1$. In the case where $|Z| > 1$, this behaviour reverses and the entanglement reaches a local maximum for $q = 1$ (i.e., non-deformed coherent states). A minimum entanglement exists in this case, however, the value of this minimum entanglement and the value of q at which it is attained are j-dependent.

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